31/B - Practice Midterm 1 - Solutions

October 12, 2011

1.a. (10 points) Let $f(x) = x^2 + 4x + 1$, and let $g(x)$ be the inverse of $f(x)$, defined on $[-2, \infty)$. Compute $g'(6)$.

Solution We know that

$$g'(6) = \frac{1}{f'(g(6))}.$$

The number $g(6)$ is a number x such that $x^2 + 4x + 1 = 6$. Solving for x, we see that $x = 1$ given the domain of $g(x)$. Since $f'(x) = 2x + 4$, we have

$$g'(6) = \frac{1}{f'(1)} = \frac{1}{2+4} = \frac{1}{6}.$$

1.b. (10 points) Compute the derivative of $\cos^{-1}(x)$ at $x = \frac{\sqrt{2}}{2}$.

Solution Set $g(x) = \cos^{-1}(x)$. We know that if $x = \frac{\pi}{4}$, then $\cos(x) = \frac{\sqrt{2}}{2}$. So,

$$g'\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{-\sin(\frac{\pi}{4})} = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$

2.a. (10 points) Suppose that you create an annuity with an initial investment of $P(0) = 10000$, an interest rate of $r = .1$, and a continuous withdrawl of $N = 5000$ per year. When does the annuity run out of money?

Solution We have the equation

$$P(t) = \frac{N}{r} + \left(P(0) - \frac{N}{r}\right) e^{rt}.$$

Solving for $P(t) = 0$, we find

$$0 = \frac{5000}{.1} + \left(10000 - \frac{5000}{.1}\right) e^{.1t} = 50000 - 40000e^{.1t}.$$

So, we run out of money at

$$t = 10 \ln \frac{5}{4}.$$

2.b. (10 points) What is the minimal initial investment so that the annuity never runs out of money?

Solution We just need $P(0) - \frac{N}{r}$ to be non-negative. So, we need to invest at least

$$\frac{N}{r} = \frac{5000}{.1} = 50000.$$

3. (20 points) Compute

$$\lim_{t \to \infty} \frac{\ln(t+2)}{\log_2 t}.$$

Solution First, we write $\log_2 t = \frac{\ln t}{\ln 2}$. Then,

$$\begin{aligned}
\lim_{t \to \infty} \frac{\ln(t+2)}{\log_2 t} &= \lim_{t \to \infty} \frac{\ln(t+2)}{\frac{\ln t}{\ln 2}} \\
&= \ln 2 \lim_{t \to \infty} \frac{\ln(t+2)}{\ln t} \\
&= \ln 2 \lim_{t \to \infty} \frac{\frac{1}{t+2}}{\frac{1}{t}} \\
&= \ln 2 \lim_{t \to \infty} \frac{t}{t+2} \\
&= \ln 2 \lim_{t \to \infty} \frac{1}{1} \\
&= \ln 2 \cdot 1 = \ln 2
\end{aligned}$$

by two applications of L'Hôpital's rule.

4. (20 points) Compute the indefinite integral

$$\int 2^x \cos x \, dx$$

Solution Let's do substitution with

$$\begin{aligned}
u &= 2^x & \qquad u' &= (\ln 2) 2^x \\
v' &= \cos x & \qquad v &= \sin x.
\end{aligned}$$

Then, we get

$$\int 2^x \cos x \, dx = 2^x \sin x - (\ln 2) \int 2^x \sin x \, dx$$

Doing substitution again for the second integral, we use

$$u = 2^x \qquad u' = (\ln 2)2^x$$
$$v' = \sin x v = -\cos x.$$

So, we have

$$\begin{aligned}\int 2^x \cos x\,dx &= 2^x \sin x - (\ln 2)\int 2^x \sin x\,dx\\ &= 2^x \sin x - (\ln 2)\left(-2^x \cos x - (\ln 2)\int -2^x \cos x\,dx\right)\\ &= 2^x \sin x + (\ln 2)2^x \cos x - (\ln 2)^2 \int 2^x \cos x\,dx.\end{aligned}$$

We add $(\ln 2)^2 \int 2^x \cos x\,dx$ to both sides and obtain

$$(1 + (\ln 2)^2)\int 2^x \cos x\,dx = 2^x \sin x + (\ln 2)2^x \cos x + C,$$

or

$$\int 2^x \cos x\,dx = \frac{1}{1 + (\ln 2)^2}\left(2^x \sin x + (\ln 2)2^x \cos x\right) + C$$

5. (20 points) Compute the indefinite integral

$$\int \sqrt{x^2 + 9}\,dx$$

Solution Let's do the substitution

$$x = 3\tan\theta$$
$$dx = 3\sec^2\theta\,d\theta.$$

We obtain

$$\begin{aligned}\int \sqrt{x^2+9}\,dx &= \int \sqrt{9\tan^2\theta + 9}\sec^2\theta\,d\theta = 3\int \sqrt{\sec^2\theta}\sec^2\theta\,d\theta\\ &= \int \sec^3\theta\,d\theta.\end{aligned}$$

Since we haven't covered this integral in class, let's leave it at that. If you want, you can use the recursive formula.